Effects of cross-correlated noises on the intensity fluctuation of the single-mode laser system

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A single-mode laser model with cross-correlated additive and multiplicative noise terms is considered, and the effects of correlation between noises on the relaxation time and the intensity correlation function are studied. Using the projection operator method and taking into account the effects of the memory kernels of the intensity correlation function, the analytic expressions for the relaxation time and the correlation function are derived. Based on numerical computations, it is found that the self-correlation time and the cross-correlation time have the same effects on the single-mode laser system.

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It is known that the single-mode dye laser model with additive white noise and multiplicative white noise is often used as a prototype to investigate the laser fluctuations\textsuperscript{[1,2-5]}\textsuperscript{[11]}. Using this method, the previous studies have revealed that the consideration of additive and multiplicative noises simultaneously is of importance to deeply understand statistical properties of the single-mode laser system\textsuperscript{[6]}. In most of these existing theoretical studies, the multiplicative noise (pump noise) and the additive noise (quantum noise) are both modeled as Gaussian white noise and are treated as uncorrelated. Recently, the effects of correlation between additive and multiplicative noises on the statistical fluctuation of the single-mode laser model have attracted close attention\textsuperscript{[7-10]}. Xie et al.\textsuperscript{[11]} considered the correlation between the two noises and studied the effects of correlation intensity λ on the relaxation time Tc and the intensity correlation function C. But that study only considered that the self-correlation time τ0 and the cross-correlation time τ of the two noises are both zero. In this paper, we consider that τ0 and τ are not zero and investigate the effects of τ0 and τ on the single-mode laser system.

The complex field-amplitude $\mathbf{E}$ of the cubic model of a single-mode laser system can be described by Langevin equation (LE)\textsuperscript{[3]}

$$\frac{d\mathbf{E}}{dt} = a_0\mathbf{E} - A|\mathbf{E}|^2\mathbf{E} + \tilde{p}(t)\mathbf{E} + \tilde{q}(t),$$

where $a_0$ and $A$ are real and respectively stand for the net gain and the self-saturation coefficients, $\tilde{p}(t)$ is the pump noise and $\tilde{q}(t)$ is the quantum noise. Performing the polar coordinate transform $\mathbf{E} = xe^{i\varphi}$, Eq. (1) can be transformed into two coupling LEs for the field-amplitude $x$ and phase $\varphi$. By decoupling them, the LE of $x$ can be obtained as\textsuperscript{[12]}

$$\frac{dx}{dt} = a_0x - Ax^3 + \frac{D}{2x} + xp(t) + q(t).$$

We only consider the intensity fluctuation of the laser system. Assuming $I$ is the laser intensity ($I = x^2$), Eq. (2) is readily written for $I$ as

$$\frac{dI}{dt} = (2a_0 - AI)I + D + 2I^{1/2}q(t) + 2Ip(t).$$

The multiplicative noise $p(t)$ and the additive noise $q(t)$ are considered to be Gaussian-type noise,

$$\langle q(t)q(t') \rangle = 2D\delta(t - t'),$$

$$\langle p(t)p(t') \rangle = \frac{Q}{\tau_0}\exp(-|t - t'|/\tau_0) \rightarrow 2Q\delta(t - t'),$$

and

$$\langle p(t)q(t') \rangle = \langle q(t)p(t') \rangle = \frac{\lambda\sqrt{QD}}{\tau}\exp[-|t - t'|/\tau] \rightarrow 2\lambda\sqrt{QD}\delta(t - t'),$$

where $Q$ and $D$ are the multiplicative and additive noise intensities respectively, $\tau_0$ and $\tau$ are the self-correlation time and cross-correlation time respectively, $\lambda$ is the intensity of correlation between $p(t)$ and $q(t)$. Applying the Novikov theorem\textsuperscript{[13]} and the Fox’s approach\textsuperscript{[14]}, the approximate Fokker-Planck equation corresponding to Eq. (3) reads

$$\frac{\partial P(I,t)}{\partial t} = L_{FP}P(I,t),$$

with

$$L_{FP} = -\frac{\partial}{\partial I}F(I) + \frac{\partial^2}{\partial I^2}B(I).$$

The drift coefficient $F(I)$ and the diffusion coefficient $B(I)$ are given by

$$F(I) = 2\left(a_0 + \frac{Q}{1 + 2a_0\tau_0} - AI\right)I + \frac{3\lambda\sqrt{QD}}{1 + 2a_0\tau}I^{1/2} + 2D,$$

$$B(I) = 2\frac{Q}{1 + 2a_0\tau_0}I^2 + \frac{\lambda\sqrt{QD}}{1 + 2a_0\tau}I^{3/2} + 2DI.$$

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It should be pointed out that the above approximate Fokker-Planck equation is valid only for the case of $1 + 2a_0\tau_0 > 0$ and $1 + 2a_0\tau > 0$. The steady-state probability density function $P_{st}(I)$ can be obtained directly from Eq. (7) as

$$P_{st}(I) = N(I + 2\alpha_0\tau_0)\sqrt{\frac{D}{Q}} \exp\left(\frac{\alpha_1(\lambda)}{Q}\right) \frac{\alpha_2(\lambda)}{Q},$$

where

$$\alpha_1(\lambda) = \frac{a_0(1 + 2a_0\tau_0)}{2} + \frac{DA(1 + 2a_0\tau_0)^3}{2Q^2} - \frac{2AD\lambda^2(1 + 2a_0\tau_0)}{Q^2(1 + 2a_0\tau)^2} - \frac{1}{2},$$

$$\alpha_2(\lambda) = \frac{\alpha_3(\lambda)}{\alpha_4(\lambda)},$$

$$\times \arctan \frac{2\sqrt{I} + 2\alpha_0\lambda_0\sqrt{D}/((1 + 2a_0\tau_0)\sqrt{Q})}{\alpha_4(\lambda)}.$$

In terms of the adjoint operator $L_{FP}^+$ of the operator given by Eq. (8), $\delta I(t + \Delta t)$ can be expressed as $\delta I(t + \Delta t) = \exp(L_{FP}^+\Delta t)\delta I(t)$. Thus one can rewrite Eq. (18) and get the associated Laplace transform

$$\hat{\tilde{C}}(\omega) = \int_0^\infty \exp(-\omega\Delta t)C(\Delta t)d\Delta t = \frac{1}{\langle(\delta I)^2\rangle_{st}} \left\langle \delta I - \frac{1}{\omega - \omega_{FP}^+}\delta I \right\rangle_{st},$$

Using the projection operator method used by Fujisaka and Grossmann[16] to deal with the Laplace resolvent $\omega - L_{FP}^+$ in Eq. (20), we have the following continued fraction expression[15, 16],

$$\hat{\tilde{C}}(\omega) = \frac{\omega + \mu_0}{\omega + \mu_1 + \frac{\eta_1}{\omega + \mu_2 + \frac{\eta_2}{\omega + \mu_3 + \cdots}}},$$

in which

$$\mu_1 = -\frac{\langle(\delta I_1)^2\rangle_{st}}{\langle(\delta I_1)^2\rangle_{st}},$$

$$\eta_1 = -\frac{\langle(\delta I_1)^2\rangle_{st}}{\langle(\delta I_1)^2\rangle_{st}},$$

$$\delta I_{i+1} = J_{i+1}L_{FP}^+\delta I_i,$$

with starting $\delta I_0 = \delta I$ and $J_0 = 1$. The operator $J_i$ is determined by

$$S_{i-1} = J_{i-1} - J_i = \frac{\langle(\delta I_1)^2\rangle_{st}}{\langle(\delta I_1)^2\rangle_{st}},$$

where the operator $\langle\delta I_1\rangle$ acting on $\varphi(I)$ means the scalar product

$$\langle\delta I_1| \varphi(I)\rangle = \langle(\delta I_1\varphi(I))\rangle_{st} = \int P_{st}(I)\delta I\varphi(I)dI,$$

where the projection operator $S_i$ projects $\varphi(I)$ onto the subspace associated with the variable $\delta I_i$, the projector $J_i$ projects onto the space orthogonal to the space containing $\delta I_i$. The basic idea behind the method used to lead a continuous fraction expansion is to identify $\delta I_i$ as a slow variable and in $J_i$ space it slaves the remaining fast variables[17]. Fujisaka et al.[16] have pointed out that earlier experience with the Duffing oscillator or with the laser fluctuation has shown that the effects of higher orders of memory are not significant. Setting $\eta_2 = 0$, the first-order approximation of $T_c$ is

$$T_c = \left[\mu_0 + \frac{\eta_1}{\mu_1}\right]^{-1},$$

and the first-order approximation of $\hat{\tilde{C}}(\omega)$ is

$$\hat{\tilde{C}}(\omega) = \frac{\omega + \mu_1}{(\omega + \mu_0)(\omega + \mu_1) + \eta_1},$$

where

$$\mu_0 = \frac{\langle B(I)\rangle_{st}}{\langle(\delta I)^2\rangle_{st}},$$

$$\eta_1 = \frac{\langle G(I)F'(I)\rangle_{st}}{\langle(\delta I)^2\rangle_{st} + \mu_0},$$
$$\mu_1 = -\frac{\langle G(I) [F'(I)]^2 \rangle_{st}}{\eta_1 \langle (\delta I)^2 \rangle_{st}} + \frac{\mu_0^2}{\eta_1} - 2\mu_0. \quad (31)$$

Performing the Laplace converse transformation of Eq. (28), we get

$$C(\Delta t) = (1 + \Delta) \exp(-\alpha_- \Delta t) + \Delta \exp(-\alpha_+ \Delta t), \quad (32)$$

where

$$\Delta = \frac{\mu_1 - \alpha_-}{\alpha_+ - \alpha_-}. \quad (33)$$

$$\alpha_\pm = \frac{1}{2} \{ \mu_0 + \mu_1 \pm [((\mu_1 - \mu_0)^2 - 4\eta_1]^{1/2} \}. \quad (34)$$

Let $\tau_0 = 0$ and $\tau = 0$, the above results fall back to Eqs. (18)–(20) presented in Ref. [11].

Using Eqs. (29)–(31), we can calculate $\mu_0$, $\eta_1$, and $\mu_1$. The effects of $\tau_0$ and $\tau$ on $T_c$ (Eq. (27)) and the intensity correlation function (Eq. (32)) of the single-mode laser system can be analyzed by the numerical calculation. We only consider the positive correlation ($\lambda > 0$) in this paper. The results are plotted in Figs. 1–3 respectively.

Figure 1 shows that the curves of $T_c$ as functions of the net gain $a_0$ with different values of $\tau_0$ (Fig. 1(a)) and $\tau$ (Fig. 1(b)). We can find that $T_c$ exhibits a single-peak with the increase of $a_0$. This means that with the increase of $a_0$, the decay rate of the intensity fluctuation in the stationary state turns over, from slowing down to speeding up. This conclusion is the same as the result in Ref. [11]. The difference only lies in the position of peak. Owning to the effects of $\tau_0$ and $\tau$, the position of peak is not near the threshold [11] ($a_0 = 0$) but far above the threshold. It is clear that $\tau_0$ and $\tau$ have the same effects on $T_c$, i.e., the larger $\tau_0$ or $\tau$ is, the larger $T_c$ is.

Figure 2 shows that the curves of $C$ as functions of $a_0$

![Fig. 2. Intensity correlation function $C$ as a function of the net gain $a_0$. (a) $Q = 1.78$, $A = 1$, $D = 2$, $\lambda = 0.8$, and $\tau = 0.5$, $\tau_0$ takes 0.3, 0.4, 0.5, respectively; (b) $Q = 1.78$, $A = 1$, $D = 2$, $\lambda = 0.8$, and $\tau_0 = 0.3$, $\tau$ takes 0.1, 0.3, 0.5, respectively.](image)

Fig. 1. Relaxation time $T_c$ as a function of the net gain $a_0$. (a) $Q = 1.78$, $D = 2$, $\lambda = 0.8$, and $\tau = 0.3$, $\tau_0$ takes 0.3, 0.4, 0.5, respectively; (b) $Q = 1.78$, $D = 2$, $\lambda = 0.8$, and $\tau_0 = 0.3$, $\tau$ takes 0.1, 0.3, 0.5, respectively.

![Fig. 3. Intensity correlation function $C$ as a function of time interval $\Delta t$. (a) $Q = 1.78$, $A = 1$, $D = 2$, $\lambda = 0.5$, and $\tau = 0.5$, $\tau_0$ takes 0.1, 0.5, 1.5, respectively; (b) $Q = 1.78$, $A = 1$, $D = 2$, $\lambda = 0.5$, and $\tau_0 = 0.5$, $\tau$ takes 0.1, 0.3, 0.9, respectively.](image)
with different values of $\tau_0$ (Fig. 2(a)) and $\tau$ (Fig. 2(b)).

It is known that $C$ is a measure of correlation between intensities at time $t$ and $t + \Delta t$. As shown, the curves exhibit a peak with the increase of $a_0$. Far above the threshold, the correlation becomes weaker and weaker with the increase of $a_0$. In Fig. 2, it is obvious that $\tau_0$ ($\tau$) plays a positive role in $C$, i.e., the larger $\tau_0$ ($\tau$) is, the larger $C$ is. From Figs. 1 and 2, we can find that $\tau_0$ and $\tau$ have the same effects on the single-mode laser system, with the increase of correlation time, the decay rate of $C$ becomes slower and slower.

As might be excepted, this result that $C$ decays with time interval $\Delta t$ (see Fig. 3), is the same as that in Ref. [11]. The effects of $\tau_0$ and $\tau$ on $C$ are exhibited in Fig. 3 too. That is, the larger $\tau_0$ or $\tau$ is, the larger $C$ is.

In conclusion, $\tau_0$ and $\tau$ have the same effects on $T_c$ and $C$. $T_c$ and $C$ increase with the increase of $\tau_0$ or $\tau$ in the case of positive correlation ($\lambda > 0$). With the increase of $a_0$, $T_c$ and $C$ have a peak, the result of which is the same as that in Ref. [11]. Owing to the effects of $\tau_0$ and $\tau$, the position of peak is far above the threshold.

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References